Continuous inclusion property, local intersection property and semicontinuous functions - prelude to the analysis of the existence of equilibrium in financial models.

Anna Serwatka

11.10.2023

Abstract

To consider statements about market equilibrium and the absence of arbitrage, we need theorems about fixed points. Various theorems about fixed points are used in proofs within financial mathematics, differing in their assumptions, such as the Mas-Colell's or Kakutani's fixed-point theorem (a generalized version of Brouwer's fixed point theorem). To comprehend all the assumptions in these theorems, an understanding of the definition of a semi-continuous function and concepts like the local intersection property and continuous conclusion property is required. During this presentation, we will introduce the relevant definitions and theorems, which will serve as a prelude to the analysis of statements concerning the existence of equilibrium in financial models.

1 Notation and definitions

Definition 1.1 *Let X, Y be topological spaces. The correspondence $T : X \to 2^Y$ has the local intersection property if $x \in X$ with $T(x) \neq \emptyset$ implies the existence of an open neighborhood $V(x)$ of x such that $\bigcap_{z \in V(x)} T(z) \neq \emptyset$.*

If Ω is a non-empty set, we say that the operator $T : \Omega \times X \to 2^Y$ has local intersection property if, for each $\omega \in \Omega, T(\omega, \cdot)$ has the local intersection property.

Definition 1.2 *A correspondence ψ from X to Y is said to have the continuous inclusion property at x if there exists an open neighborhood O_x of x and a nonempty correspondence $F_x : O_x \to 2^Y$ such that $F_x(z) \subseteq \psi(z)$ for any $z \in O_x$ and $\operatorname{co} F_x$ has a closed graph.*

Let (X, d) be a metric space, C be a non-empty subset of X and $T : C \to 2^X$ be a correspondence.

We will use the following notations. We denote by $B(x,r) = \{y \in C : d(y,x) < r\}$. If B_0 is a subset of X, then we will denote $B(B_0, r) = \{y \in C : d(y, B_0) < r\}$, where $d(y, B_0) = \inf_{x \in B_0} d(y,x)$.

Definition 1.3 *Let X be a topological space and Y be a normed linear space. The correspondence $T : X \to 2^Y$ is said to be almost lower semicontinuous (a.l.s.c.) at $x \in X$, if, for any $\varepsilon > 0$, there exists a neighborhood $U(x)$ of x such that $\bigcap_{z \in U(x)} B(T(z); \varepsilon) \neq \emptyset$.*

T is almost lower semicontinuous if it is a.l.s.c. at each $x \in X$.

If Ω is a non-empty set, we say that the operator $T : \Omega \times X \to 2^Y$ is almost lower semicontinuous if, for each $\omega \in \Omega, T(\omega, \cdot)$ is almost lower semicontinuous.

Definition 1.4 *Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence. The graph of $T : X \to 2^Y$ is the set $\mathrm{Gr}(T) := \{(x,y) \in X \times Y : y \in T(x)\}$.$T$ is said to be upper semicontinuous if, for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$.*

Definition 1.5 *Let X, Y be topological spaces and $T : X \to 2^Y$ be a correspondence. T is said to be lower semicontinuous if, for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$.*